Thermal Marginal Instability of Magnetohydrodynamic Micropolar Fluid Layer Heated from Below

Bhupander Singh
Professor
Department of Mathematics
Meerut College, Meerut
Email: bhupandersingh1969@yahoo.com

Reference to this paper should be made as follows:

Bhupander Singh

"Thermal Marginal Instability of Magnetohydrodynamic Micropolar Fluid Layer Heated from Below"

Voyager: Vol. XIV, 2023
Article No. 08
pp.58-69

Online available at:
https://anubooks.com/journal-volume/voyager-vol-xiv-2022--208

DOI: https://doi.org/10.31995 voyager.2023.v14.008

Abstract

Thermal marginal instability of magnetohydrodynamic micropolar fluid layer heated from below has been investigated. The momentum, angular momentum, energy and maxwell equations are solved by normal mode analysis with free boundaries of the layer. The marginal instability effects of the pertinent parameters like magnetic field, coupling parameter, micropolar heat conduction parameter and micropolar coefficient are observed and predicted by the graphs.

Keywords

Marginal Instability, Magnetohydrodynamic, Micropolar fluid, Coupling parameter and Normal Mode analysis.

https://doi.org/10.31995/voyager.2023.v14.008

Introduction

Thermal instability of a fluid layer heated from below has been studied by many researchers. Bénard [1] in 1900 did an experiment of a fluid layer heated from below and observed a thermal instability. The theoretical analysis of Bénard's experiment has been studied by Rayleigh [2] and this analysis has also received a considerable importance due to its relevance in various fields such as chemical and industrial engineering, soil mechanics, geophysics etc. The main objectives of the various studies related to the thermal instability, in particular, is to determine the critical Rayleigh number at which the onset of instability sets in either as stationary convection or through oscillations.

The Rayleigh-Bénard convection in micropolar fluids heated from below has been extensively studied by Ahmadi [3], Datta and Sastry [4], Bhattacharyya and Jena [7], L.E. Payne and B. Straughan [5]. The common results of all these studies are found that the stationary convection is the preferred mode of instability and the microrotation has a stability effect on the onset of Rayleigh-Bénard convection. An excellent review as well as large number of new developments are given by Chandrasekhar [6] in his celebrated book on hydrodynamic and hydromagnetic stability. In these methods of stability study a linear theory is usually employed *i.e.*, the equations governing the disturbances are linearized and then the grow or decay of the disturbances is studied. The effect of a magnetic field on the onset of convection in a horizontal micropolar fluid layer heated from below has also been investigated by several researchers. The extension of micropolar flows to include magneto-hydrodynamics effects is of interest in regard to various engineering applications such as in the design of the cooling systems for nuclear reactors, MHD electrical power generation, shock tubes, pump, flow meters etc. The effects of throughflow and magnetic field on the onset of Bénard convection in a horizontal layer of micropolar fluid confined between two rigid, isothermal and microrotation free, boundaries have been studied by Narasimha Murty [9]. Z Alloui and P. Vasseur [10] studied onset of Rayleigh-Bénard MHD convection in a micropolar fluid.

R.C. Sharma and P. Kumar [8] studied the effect of magnetic field on micropolar fluids heated from below and they also studied the effects of magnetic field on micropolar fluids heated from below in porous medium. In both papers, they found that in the presence of various coupling parameters, magnetic field has a stabilizing effect on stationary convection.

In this paper, I studied thermal instability of magnetohydrodynamic micropolar fluid layer and it is found that the Chandrasekhar number has a significant role in the investigation of the nature of magnetic field. To the best of my knowledge this problem is investigated so far.

Mathematical Modelling

Consider a two dimensional horizontal, electrically non-conducting, incompressible micropolar fluid layer of thickness *d*. This layer is heated from below such that the lower boundary is held at constant temperature $T = T_0$ and the upper boundary is held at fixed temperature $T = T_1$ so that a uniform temperature gradient $\beta = \left|\frac{dT}{dz}\right|$ is maintained. The physical geometry is one of infinite extent in *x* and *y* directions bounded by the planes $z = 0$ and $z = d$. The whole system is acted on by gravity force $\vec{\mathbf{g}}(0, 0, -g)$.

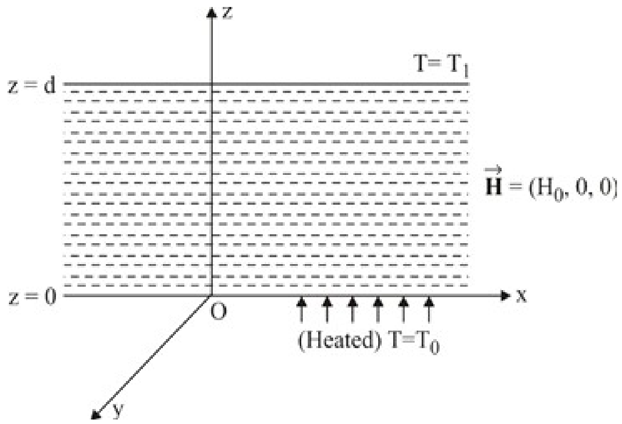

Fig. 1

A uniform magnetic field $\vec{\mathbf{H}} = (H_0, 0, 0)$ is applied along *x*-direction and the magnetic Reynolds number is assumed to be small, so that the induced magnetic field can be neglected in comparison with the applied magnetic field.

Using Boussinesq approximation, the governing equations of the problem describe the as follows:

$$\nabla.\vec{\mathbf{q}} = 0 \tag{1}$$

$$\rho_0\left[\frac{\partial\vec{\mathbf{q}}}{\partial t} + (\vec{\mathbf{q}}.\nabla)\vec{\mathbf{q}}\right] = -\nabla p - \rho g\,\hat{e}_z + (\mu+\zeta)\nabla^2\vec{\mathbf{q}} + \zeta\nabla\times\vec{\mathbf{N}} + \frac{\mu_e}{4\pi}(\nabla\times\vec{\mathbf{H}})\times\vec{\mathbf{H}} \tag{2}$$

$$\rho_0\, j\left[\frac{\partial\vec{\mathbf{N}}}{\partial t} + (\vec{\mathbf{q}}.\nabla)\vec{\mathbf{N}}\right] = (\alpha'+\beta')\,\nabla(\nabla.\vec{\mathbf{N}}) + \gamma'\nabla^2\vec{\mathbf{N}} + \zeta(\nabla\times\vec{\mathbf{q}} - 2\vec{\mathbf{N}}) \tag{3}$$

$$\rho_0 C_v\left[\frac{\partial T}{\partial t} + (\vec{\mathbf{q}}.\nabla)T\right] = \chi_T\,\nabla^2 T + \delta(\nabla\times\vec{\mathbf{N}}).\nabla T \tag{4}$$

$$\rho = \rho_0[1 - \alpha(T - T_0)] \tag{5}$$

$$\frac{\partial \vec{\mathbf{H}}}{\partial t} = \nabla \times (\vec{\mathbf{q}} \times \vec{\mathbf{H}}) + \gamma_m \nabla^2 \vec{\mathbf{H}} \tag{6}$$

$$\nabla.\vec{\mathbf{H}} = 0 \tag{7}$$

Where $\vec{\mathbf{q}}$, $\vec{\mathbf{N}}$, p, ρ, ρ_0, μ, ζ, μ_e, j, α', β', γ', C_v, T, t, χ_T, δ, α, T_0, $\hat{\mathbf{e}}_z$ and γ_m denote respectively fluid velocity, microrotation, pressure, fluid density, reference density, fluid viscosity, coupling viscosity coefficient, magnetic permeability, microinertia coefficient, micropolar viscosity coefficients, specific heat at constant volume, temperature, time, thermal conductivity, micropolar heat conduction coefficient, coefficient of thermal expansion, reference temperature, unit vector along z-direction and the magnetic viscosity.

Perturbation Solutions

Using basic state $\vec{\mathbf{q}} = \vec{\mathbf{q}}_b = (0, 0, 0)$, $\vec{\mathbf{N}} = \vec{\mathbf{N}}_b = (0, 0, 0)$, $p = p_b(z)$, $\rho = \rho_b(z)$ and $\vec{\mathbf{H}} = \vec{\mathbf{H}}_b = (H_0, 0, 0)$

and perturbations $\boldsymbol{q}', \boldsymbol{N}', \boldsymbol{p}', \boldsymbol{h}, \boldsymbol{\theta}$ in $\boldsymbol{q}, \boldsymbol{N}, \boldsymbol{p}, \boldsymbol{H}, \boldsymbol{T}$, the governing equations (1) to (7) in linear form become

$$\rho_0 \frac{\partial \vec{\mathbf{q}}'}{\partial t} = -\nabla p' + \rho_0 g \alpha \theta \hat{\mathbf{e}}_z + (\mu + \zeta)\nabla^2 \vec{\mathbf{q}}' + \zeta \nabla \times \vec{\mathbf{N}}' + \frac{\mu_e}{4\pi}(\nabla \times \vec{\mathbf{h}}) \times \vec{\mathbf{H}}_b \tag{8}$$

$$\rho_0 j \frac{\partial \vec{\mathbf{N}}'}{\partial t} = (\alpha' + \beta')\nabla(\nabla.\vec{\mathbf{N}}') + \gamma' \nabla^2 \vec{\mathbf{N}}' + \zeta(\nabla \times \vec{\mathbf{q}}' - 2\vec{\mathbf{N}}') \tag{9}$$

$$\rho_o C_v \left[\frac{\partial \theta}{\partial t} + (\vec{\mathbf{q}}'.\nabla)T_b\right] = \chi_T \nabla^2 \theta + \delta(\nabla \times \vec{\mathbf{N}}').\nabla T_b \tag{10}$$

$$\frac{\partial \vec{\mathbf{h}}}{\partial t} = (\vec{\mathbf{H}}_b.\nabla)\vec{\mathbf{q}}' + \gamma_m \nabla^2 \vec{\mathbf{h}} \tag{11}$$

$$\nabla.\vec{\mathbf{h}} = 0 \tag{12}$$

Further using the following non-dimensional transformations

$$x = dx^*,\ y = dy^*,\ z = dz^*,\ \vec{\mathbf{h}} = H_0 \vec{\mathbf{h}}^*,\ \vec{\mathbf{q}}' = \frac{K_T}{d}\vec{\mathbf{q}}^*,\ t = \frac{\rho_o d^2}{\mu} t^*,\ p' = \frac{\mu K_T}{d^2} p^*$$

$\vec{\mathbf{N}}' = \dfrac{K_T}{d^2}\vec{\mathbf{N}}^*$, $\theta = \beta d \theta^*$, $K_T = \dfrac{\chi_T}{\rho_0 C_v}$, where K_T is the thermal diffusivity and ignoring the stars, the equations (8) to (12) yield

$$\frac{\partial \mathbf{q}}{\partial t} = -\nabla p + R\theta \hat{\mathbf{e}}_z + (1+K)\nabla^2 \vec{\mathbf{q}} + K\nabla \times \vec{\mathbf{N}} + Q(\nabla \times \vec{\mathbf{h}}) \times \hat{\mathbf{e}}_x \tag{13}$$

$$\bar{\mathbf{j}}\ \frac{\partial \vec{\mathbf{N}}}{\partial t} = C'\nabla\ (\nabla.\vec{\mathbf{N}}) - C\ \nabla\times\ \nabla\times\ \vec{\mathbf{N}} + K(\nabla \times \vec{\mathbf{q}} - 2\vec{\mathbf{N}}) \tag{14}$$

$$P_r \frac{\partial \theta}{\partial t} = \nabla^2\theta + W - \bar{\boldsymbol{\delta}}\xi \tag{15}$$

$$P_r \frac{\partial \vec{\mathbf{h}}}{\partial t} = \frac{\partial \vec{\mathbf{q}}}{\partial x} + \frac{P_r}{P_m}\nabla^2 h \tag{16}$$

$$\nabla.\vec{\mathbf{h}} = 0 \tag{17}$$

Where $R = \frac{\rho_0 g \alpha \beta d^4}{\mu K_T}$ is the thermal Rayleigh number, $Q = \frac{\mu_e H_0^2 d^2}{4\pi \mu K_T}$ is the Chandrasekhar number, $C' = \frac{\alpha' + \beta' + \gamma'}{\mu d^2}$, $C = \frac{\gamma'}{\mu d^2}$, $K = \frac{\zeta}{\mu}$, $\xi = (\nabla \times \vec{\mathbf{N}})z$, $W = \vec{\mathbf{q}}.\ \hat{e}_z$, $P_r = \frac{\mu}{\rho_o K_T}$ is the Prandtl number and $P_m = \frac{\mu}{\rho_o \gamma_m}$ is the magnetic Prandtl number.

Boundary Conditions

Consider that both the boundaries of the problem are free and perfectly heat conducting so that

$$w = 0 = \frac{\partial^2 w}{\partial z^2},\ \vec{\mathbf{N}} = 0,\ \theta = 0 \text{ at } z = 0 \text{ and } z = 1,\ \zeta = 0 \text{ at } z = 0, 1 \tag{18}$$

Dispersion Equations

Applying Curl operator to the equations (13) to (17), these equations reduce to

$$\frac{\partial}{\partial t}(\nabla \times \vec{\mathbf{q}}) = R\left[\frac{\partial \theta}{\partial y}\hat{e}_x - \frac{\partial \theta}{\partial x}\hat{e}_y\right] + (1+K)\nabla^2(\nabla \times \vec{\mathbf{q}}) + K\nabla \times \nabla \times \vec{\mathbf{N}} + Q\frac{\partial}{\partial x}(\nabla \times \vec{\mathbf{h}}) \tag{19}$$

$$\frac{\partial}{\partial t}\nabla^2 w = R\nabla_1^2\theta + (1+K)\nabla^4 w + K\nabla^2\xi + Q\frac{\partial}{\partial x}(\nabla^2 h_z) \tag{20}$$

$$\bar{\mathbf{j}}\ \frac{\partial \xi}{\partial t} = C\ \nabla^2\xi - K(\nabla^2 w + 2\xi) \tag{21}$$

$$\left\{\left[\bar{\mathbf{j}}\frac{\partial}{\partial t} - C\ \nabla^2 + 2K\right]\left[\frac{\partial}{\partial t} - (1+K)\nabla^2\right] + K^2\nabla^2\right\}\zeta_z = Q\left[\bar{\mathbf{j}}\frac{\partial}{\partial t} - C\nabla^2 + 2K\right]\frac{\partial m_z}{\partial x} \tag{22}$$

https://doi.org/10.31995/voyager.2023.v14.008

$$P_r \frac{\partial m_z}{\partial t} = \frac{\partial \zeta_z}{\partial x} + \frac{P_r}{P_m} \nabla^2 m_z \tag{23}$$

$$P_r \frac{\partial \theta}{\partial t} = \nabla^2 \theta + w - \bar{\boldsymbol{\delta}}\, \xi \tag{24}$$

$$P_r \frac{\partial h_z}{\partial t} = \frac{\partial w}{\partial x} + \frac{P_r}{P_m} \nabla^2 h_z \tag{25}$$

Normal Mode Analysis

Applying the following normal mode transformations

$$[w, \zeta_z, \xi, \theta, m_z, h_z] = [W(z), X(z), G(z), \Theta(z), M(z), B(z)]\ e^{i(k_x x + k_y y) + \sigma t}$$

Where $\nabla \equiv D^2 - a^2$, $\nabla_1^2 = -a^2$, with $a = \sqrt{k_x^2 + k_y^2}$ as the wave number and σ is stability parameter, to equations (19) to (25), I have

$$\left[\sigma(D^2 - a^2) - (1+K)(D^2 - a^2)^2\right] W = -Ra^2 \Theta + K(D^2 - a^2) G + ik_x Q(D^2 - a^2) B \tag{26}$$

$$\left[\bar{j}a - C(D^2 - a^2) + 2K\right] G = -K(D^2 - a^2) W \tag{27}$$

$$\left\{\left[\bar{j}\sigma - C(D^2 - a^2) + 2K\right]\left[\sigma - (1+K)(D^2 - a^2)\right] + K^2(D^2 - a^2)\right\} X = Q\, i\, k_x [\bar{j}\,\sigma - C(D^2 - a^2) + 2K] M \tag{28}$$

$$\frac{P_r}{P_m}\left[P_m \sigma - (D^2 - a^2)\right] M = ik_x X \tag{29}$$

$$\left[P_r \sigma - (D^2 - a^2)\right] \Theta = W - \bar{\boldsymbol{\delta}} G \tag{30}$$

$$\frac{P_r}{P_m}\left[P_m\ \sigma - (D^2 - a^2)\right] B = i\, k_x W \tag{31}$$

Boundary conditions (18) becomes

$$\left.\begin{array}{l} W = D^2 W = \Theta = G = 0,\ M = 0 \\ DX = 0, DM = 0, B = 0 \\ \text{at } z = 0 \text{ and } z = 1 \end{array}\right] \tag{32}$$

Using (32), equations (26) to (31) yield

$$D^4 W = 0 = D^2 B = D^2 G = D^2 M = D^2 \Theta \text{ at } z = 0 \text{ and } z = 1 \tag{33}$$

Using the boundary conditions (32) and (33), it is obtained

$D^{2n}W = 0$ at $z = 0, 1$, where n is a positive integer.

Thus, the proper solution W characterizing the lowest mode is

$$W = W_0 \sin \pi z \qquad (34)$$

Where W_0 is a constant.

Eliminating Θ, G, M, X, B between (26)-(31), the resulting equation is simplified after substituting W from (34) as

$$\left[\sigma b + (1+K)b^2\right]\left[\bar{\mathbf{j}}\,\sigma + Cb + 2K\right]\left[P_r\sigma + b\right]\frac{P_r}{P_m}\left[P_m\sigma + b\right]$$

$$= Ra^2\frac{P_r}{P_m}\left[P_m\,\sigma + b\right]\left[\bar{\mathbf{j}}\,\sigma + Cb + 2K - \bar{\boldsymbol{\delta}}Kb\right] + K^2b^2\left[P_r\sigma + b\right]\frac{P_r}{P_m}\left[P_m\sigma + b\right]$$

$$-b\,k_x^2\,Q\,[\bar{\mathbf{j}}\sigma + Cb + 2K]\,[P_r\sigma + b] \qquad (35)$$

Where $b = \pi^2 + a^2$

Marginal Instability

For the stationary marginal state setting $\sigma = 0$ in (35), I obtain

$$R = \frac{b^3[C(1+K)b + 2K + K^2] + \dfrac{bk_x^2P_m}{P_r}Q(Cb+2K)}{a^2(Cb + 2K - \bar{\boldsymbol{\delta}}Kb)} \qquad (36)$$

In the absence of magnetic field $(Q = 0)$ and coupling parameter $(\bar{\delta} = 0)$, equation (36) reduces to

$$R = \frac{b^3}{a^2}\left[\frac{b(1+K)C + 2K + K^2}{(Cb + 2K)}\right]$$

Which is the same as proposed by **[Goodarz Ahmadi]**.

In the absence of magnetic field $(Q = 0)$, the equation (36) reduces to

$$R = \frac{b^3}{a^2}\left[\frac{C(1+K)b + 2K + K^2}{(C - \bar{\boldsymbol{\delta}}K)b + 2K}\right]$$

Which is the same as proposed by [**L.E. Payne** and **B. Straughan**] and [**Y. Qin** and **P.N. Kaloni**].

Putting $A = \dfrac{K}{C}$ in equation (36), it becomes

$$R = \frac{P_r b^3[2A+b+K(A+b)]+b^2 k_x^2 P_m Q(2A+b)}{a^2 P_r(2A+b-\bar{\boldsymbol{\delta}}Ab)} \qquad (37)$$

In order to investigate the effects of magnetic field Q, coupling parameter K, micropolar heat conduction parameter $\bar{\delta}$ and micro coefficient A, we examine the behaviour of $\frac{dR}{dQ}, \frac{dR}{dK}, \frac{dR}{d\bar{\delta}}$ and $\frac{dR}{dA}$.

Differentiating both sides of (37) with respect to Q, I get

$$\frac{dR}{dQ} = \frac{b^2 k_x^2 P_m(2A+b)}{a^2 P_r(2A+b-\bar{\delta}Ab)}$$

It is observed here that $\frac{dR}{dQ}$ is always positive when $\bar{\boldsymbol{\delta}} < \frac{1}{A}$, thus, the magnetic field has stabilizing effect if $\bar{\boldsymbol{\delta}} < \frac{1}{A}$.

Differentiating both sides of (37) with respect to K, I get

$$\frac{dR}{dK} = \frac{b^3(A+b)}{a^2(2A+b-\bar{\boldsymbol{\delta}}Ab)}$$

Which is always positive if $\bar{\boldsymbol{\delta}} < \frac{1}{A}$ thus, the coupling parameter K has a stabilizing effect, if $\bar{\delta} < \frac{1}{A}$

Differentiating both sides of (37) with respect to $\bar{\delta}$, I get

$$\frac{dR}{d\bar{\delta}} = \frac{Ab\left[b^2 k_x^2 P_m Q(2A+b)+P_r b^3\{2A+b+K(A+b)\}\right]}{a^2 P_r(2A+b-\bar{\boldsymbol{\delta}}Ab)^2}$$

which is always positive, thus, the micropolar heat conduction parameter $\bar{\delta}$ has stabilizing effect.

Differentiating both sides of (37) with respect to A, I get

$$\frac{dR}{dA} = \frac{b^5\bar{\boldsymbol{\delta}}P_r(1+K)+b^4\{-P_r K+\bar{\boldsymbol{\delta}}k_x^2 P_m Q\}}{a^2 P_r(2A+b-\bar{\delta}Ab)^2}$$

Now $\frac{dR}{dA} > 0$ if $Q > \frac{P_r K}{\bar{\boldsymbol{\delta}} k_x^2 P_m}$

Thus, the micropolar coefficient A has stabilizing effect if $Q > \frac{P_r K}{\bar{\boldsymbol{\delta}} k_x^2 P_m}$.

In the absence of $\bar{\boldsymbol{\delta}}(\bar{\boldsymbol{\delta}} = 0)$, $\frac{dR}{dA} < 0$, which predicts that the micropolar coefficient *A* has destabilizing effect.

Results and Conclusions

1. The magnetic field has stabilizing effect when $\bar{\delta} < \frac{1}{A}$
2. The coupling parameters *K* has a stabilizing effect if $\bar{\delta} < \frac{1}{A}$
3. The micropolar heat conduction parameter $\bar{\delta}$ has stabilizing effect.
4. The micropolar coefficient *A* has stabilizing effect when $Q > \frac{P_r K}{\bar{\delta} k_x^2 P_m}$
5. In the absence of micropolar heat conduction parameter, the micropolar coefficient has destabilizing effect.

Graphical Representation

With the help of excel programming the variation of thermal Rayleigh number with respect to Magnetic field (Q), Coupling parameter (K), Micropolar heat conduction parameter ($\bar{\delta}$), Micropolar coefficient (A) are shown by the following graphs:

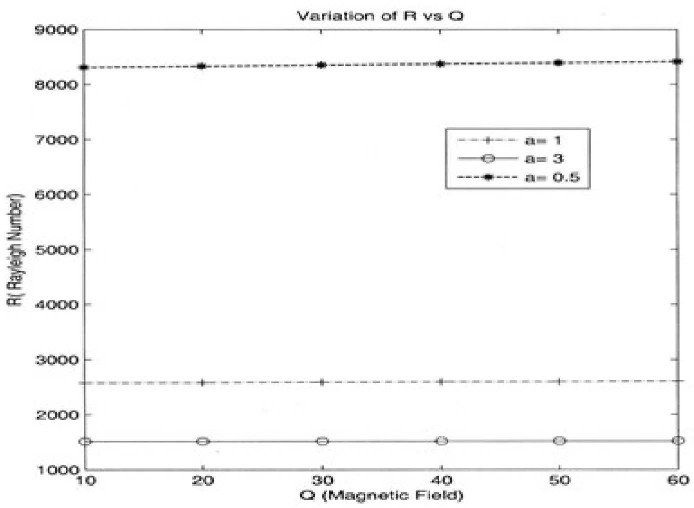

Fig. 2: Marginal instability curve for the variation of R vs Q for A=0.5, K=1, P_r=2, P_m=4, $\bar{\delta}$= 0.5, k_x=0.5

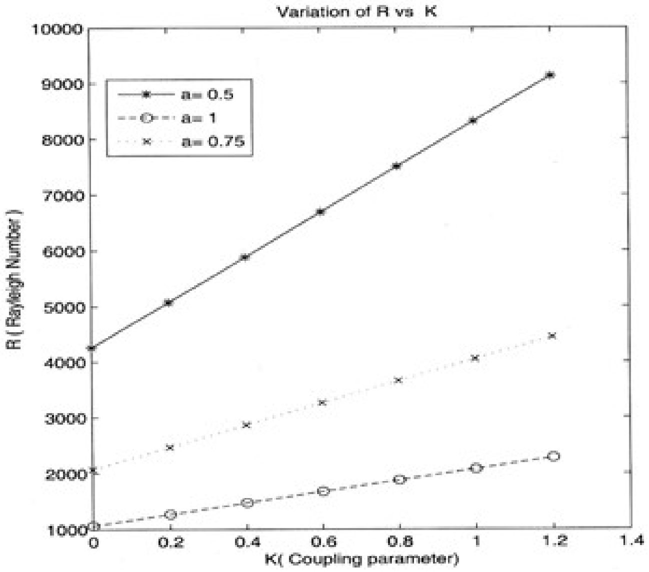

Fig. 3: Marginal instability curve for the variation of R vs K for A=0.5, P_r=2, P_m=4, Q=10, =0.05, = 0.5.

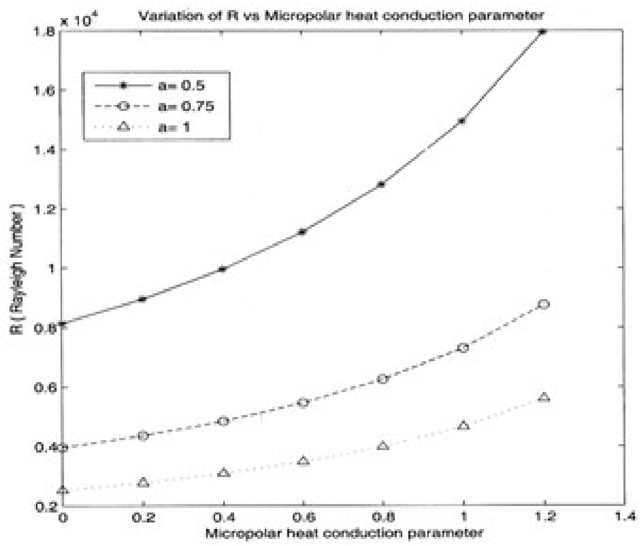

Fig. 4: Marginal instability curve for the variation of R vs for A=0.5, P_r=2, P_m=4, K=1, =0.05, Q=20.

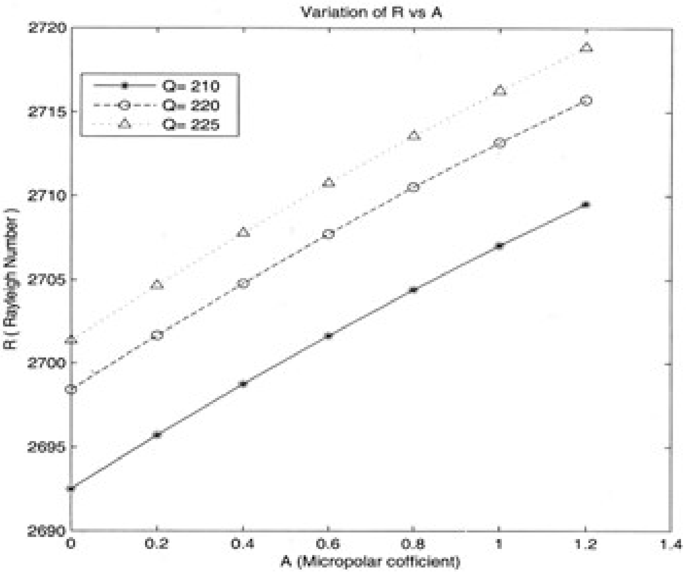

Fig. 5: Marginal instability curve for the variation of R vs A for a=1, P_r=2, P_m=4, K=1, =0.05, =0.05

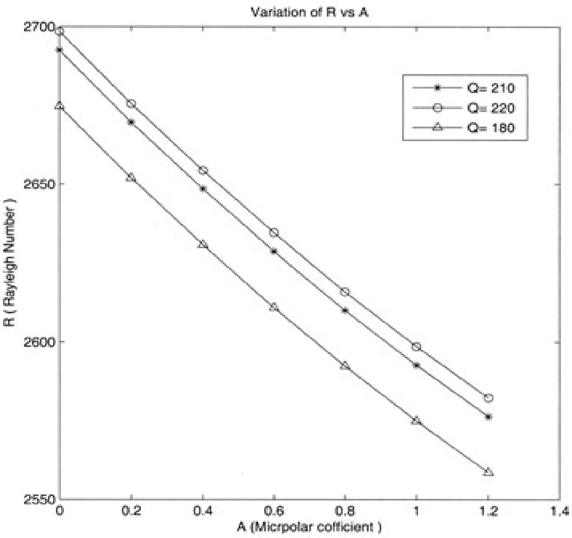

Fig. 6: Marginal instability curve for the variation of R vs A for =0, P_r=2, P_m=4, K=1, =0.05, a=1.

https://doi.org/10.31995/voyager.2023.v14.008

References

1. Benard, H. (1900). Revue Generale des Sciences Pures at Appliqués. 11. 1261.
2. Rayleigh, L. (1962). On convection currents in a horizontal layer of fluid, when the higher temperature is on the underside. Phil. *Mag.* vol. 32(6). Pg. **529.** (1916a). Also scientific papers: 6. 432. (1920). Reprinted by Saltzman, (1962).
3. Ahmadi, G. (1976). Stability of micropolar fluid layer heated from below. *Int. Engg. Sci.* vol. 14. Pg. **81-89.**
4. Datta, A.B., Shastry, V.U.K. (1976). Thermal instability of a horizontal layer of micropolar fluid heated from below. *Int. J. Engg. Sci.* vol. 14. Pg. **631-637**.
5. Payne, L.E., Straughan, B. (1989). *Int. J. Eng. Sci*. 27. 827.
6. Chandrasekhar, S. (1981). *Hydrodynamic and Hydromagnetic Stability*. Dover publications: New York.
7. Battacharyya, S.P., Jena, S.K. (1983). *Int. Eng. Sci*. 21. 1019.
8. Sharma, R.C., Kumar, P. (1995). On micropolar fluids heated from below in hydromagnetics. *J. Non-Equilibrium Thermodynamics*. vol. 20. Pg. **150-159.**
9. Murty, Y. Narasimha. (2001). Effect of throughflow and magnetic field on Bénard convection in micropolar fluid. *Acta Mech*. 150. 11.
10. Alloui, Z., Vasseur, P. (2012). Onset of Rayleigh-Bénard MHD convection in a micropolar fluid. *Int. J. Heat Mass Transfer.* 55. 1164.